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Thermodynamics of the standard quantum harmonic oscillator of time-dependent frequency with and without inverse quadratic potential

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Abstract
Making use of a dynamical invariant operator, we obtain a quantum mechanical solution of a damped harmonic oscillator having time-dependent frequency with and without inverse quadratic potential. We confirm that the uncertainty relation is always larger than $\hbar/2$ in both the number and thermal state. We obtain a density operator satisfying the Liouville–von Neumann equation, and use its diagonal elements to calculate various expectation values in the thermal state.

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1. Introduction

Undoubtedly, vibration is one of the most dominant physical phenomena that we meet in everyday life and for small oscillation it can be approximated to harmonic oscillator motion. Since the birth of quantum physics, the investigation of the harmonic oscillator has contributed to the development of theoretical physics [1]. The quantum mechanical problems of explicitly time-dependent harmonic oscillators have been of great interest in the literature of physics.

The problem of the quantum oscillator with time-dependent frequency has been treated with other methods earlier by Feynman [2] and Schwinger [3]. The same problem in terms of coherent states was considered in detail in the book by Perelomov [4].

There are three kinds of methods to find the quantum solution of a time-dependent harmonic oscillator: the propagator method [5–7], the invariant operator method [8–13] and the unitary transformation method [14–17]. We use the invariant operator method and the unitary transformation method together to evolve the quantum and statistical theory of a damped harmonic oscillator of time-dependent frequency with and without inverse quadratic potential. Since the introduction of the invariant quantity by Lewis [18, 19], the relation between time-dependent harmonic oscillator and its dynamical invariant operator has been fully

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discussed in the literature [20–22]. The quantum solution of the harmonic oscillator with time-dependent frequency and inverse quadratic potential is treated by Kaushal and Parashar [23]. Recently, invariant quantities for three-dimensional time-dependent Hamiltonians have been found that are applicable to a system of Coulomb interacting particles subject to a time-dependent quadratic potential [24].

The Liouville–von Neumann approach to nonequilibrium dynamics has been introduced in the literature [25], and is applicable to both the time-dependent harmonic and unharmonic oscillator. The density operator of the system obtained by taking advantage of the Schrödinger solution can be used to calculate various expectation values of the thermal state.

In section 2, we investigate the quantum mechanical solution and thermal properties of the damped harmonic oscillator with time-dependent frequency. A similar explanation of damped harmonic oscillator with time-dependent frequency and inverse quadratic potential is contained in section 3. Finally, section 4 summarizes this paper, with conclusions about the physical results of the system.

2. Harmonic oscillator of inversely decreasing frequency

The Hamiltonian for the damped harmonic oscillator that has time-dependent frequency is given by

$$\hat{H} = \mathrm{e}^{-2\beta t}\frac{\hat{p}^2}{2m} + \frac{1}{2}\mathrm{e}^{2\beta t}m\omega^2(t)\,\hat{x}^2, \tag{1}$$

where β is the damping constant. We consider the case that the frequency decreases inversely with time. Then, $\omega(t)$ can be written as

$$\omega(t) = \frac{\omega_0}{t}, \tag{2}$$

where ω_0 is positive constant with no dimension. The classical equation of motion satisfying equation (1) is

$$\ddot{\hat{x}} + 2\beta\dot{\hat{x}} + \left(\frac{\omega_0}{t}\right)^2\hat{x} = 0. \tag{3}$$

We only consider the underdamped case limited by $0 < t < \omega_0/\beta$. Then, the classical solution of the above equation is

$$\hat{x}(t) = \mathrm{e}^{-\beta t}t^{1/2}[C_+\,I_\nu(\beta t) + C_-\,I_{-\nu}\,(\beta t)], \tag{4}$$

where C_+ and C_- are integral constants, $I_\nu(z)$ is the modified Bessel function of the first kind and subscript ν is given by

$$\nu = \tfrac{1}{2}\sqrt{1-4\omega_0^2}. \tag{5}$$

From the definition of invariant operator, we can easily confirm that the invariant operator, $\hat{I}$, must satisfy the following relation:

$$\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} = \frac{\partial\hat{I}}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{I},\ \hat{H}] = 0. \tag{6}$$

Let us write the trial invariant operator in quadratic form:

$$\hat{I}(t) = A_1(t)\hat{p}^2 + A_2(t)(\hat{x}\hat{p} + \hat{p}\hat{x}) + A_3(t)\hat{x}^2, \tag{7}$$

where the coefficients $A_1(t) \sim A_3(t)$ must be determined. Substitution of equations (1) and (7) into equation (6) gives them. Then, we can obtain the full invariant operator as

$$\hat{I}(t) = \frac{1}{2\Omega}\left\{\frac{\Omega^2}{s^2(t)}\hat{x}^2 + [m\mathrm{e}^{2\beta t}\dot{s}(t)\,\hat{x} - s(t)\hat{p}]^2\right\}, \tag{8}$$

where $s(t)$ and Ω are given as, respectively,

$$s(t) = \mathrm{e}^{-\beta t} t^{1/2} [c_1 I_\nu^2(\beta t) + c_2 I_\nu(\beta t) I_{-\nu}(\beta t) + c_3 I_{-\nu}^2(\beta t)]^{1/2}, \tag{9}$$

$$\begin{aligned}\Omega^2 = {} & \frac{4c_1c_3 - c_2^2}{4} m^2 t^2 \\ & \times \left\{ I_\nu(\beta t) \left[\left(\frac{1}{2t} - \beta \right) I_{-\nu}(\beta t) + \frac{\beta}{2} (I_{-\nu-1}(\beta t) + I_{-\nu+1}(\beta t)) \right] \right. \\ & \left. - I_{-\nu}(\beta t) \left[\left(\frac{1}{2t} - \beta \right) I_\nu(\beta t) + \frac{\beta}{2} (I_{\nu-1}(\beta t) + I_{\nu+1}(\beta t)) \right] \right\}^2 .\end{aligned} \tag{10}$$

We can check that these satisfy the following differential equations, respectively, by direct differentiation:

$$\ddot{s} + 2\beta \dot{s} + \left(\frac{\omega_0}{t} \right)^2 s - \frac{\Omega^2}{m^2 s^3} \mathrm{e}^{-4\beta t} = 0, \tag{11}$$

$$\dot{\Omega} = 0. \tag{12}$$

From equation (12) we can confirm that Ω is a constant of motion. The eigenvalue equation of $\hat{I}$ can be written as

$$\hat{I}|n\rangle = \lambda_n |n\rangle. \tag{13}$$

Equation (13) with equation (8) cannot be solved easily, as it is very complicated. Therefore, to simplify the problem, we perform unitary transformation of it as

$$\hat{I}' = \hat{U}^{-1} \hat{I} \hat{U}, \tag{14}$$

where the unitary operator $\hat{U}$ is given by

$$\hat{U} = \exp \left[\mathrm{i} m \mathrm{e}^{2\beta t} \frac{\dot{s}}{2\hbar s} \hat{x}^2 \right]. \tag{15}$$

Then, the invariant operator may be converted simply to

$$\hat{I}' = -\frac{\hbar^2 s^2}{2\Omega} \frac{\partial^2}{\partial \hat{x}^2} + \frac{\Omega}{2s^2} \hat{x}^2. \tag{16}$$

We can write the eigenvalue equation of I' as

$$\hat{I}'|n'\rangle = \lambda_n |n'\rangle. \tag{17}$$

For more simplification, we introduce the variable $\hat{X}$:

$$\hat{X} = \frac{\sqrt{\Omega}}{s} \hat{x}. \tag{18}$$

Then, equation (16) can be expressed with this variable as

$$\hat{I}' = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial \hat{X}^2} + \frac{1}{2} \hat{X}^2, \tag{19}$$

and the eigenvalue equation of this has the following form:

$$\hat{I}'|N'\rangle = \lambda_n |N'\rangle, \tag{20}$$

where $|N'\rangle$ is related with $|n'\rangle$ by

$$|N'\rangle = \sqrt[4]{\frac{s^2}{\Omega}} \, |n'\rangle. \tag{21}$$

Since equation (19) is of the same form as the Hamiltonian of the ordinary simple harmonic oscillator with unit mass and frequency, we can easily identify its eigenstate and eigenvalue as, respectively,

$$\langle \hat{X}|N'\rangle = \sqrt[4]{\frac{1}{\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{1}{\sqrt{\hbar}}\hat{X}\right)\exp\left(-\frac{1}{2\hbar}\hat{X}^2\right), \tag{22}$$

$$\lambda_n = \hbar(n+\tfrac{1}{2}). \tag{23}$$

Using equations (18) and (21), equation (22) can be converted in terms of $\hat{x}$ as

$$\langle \hat{x}|n'\rangle = \sqrt[4]{\frac{\Omega}{\hbar\pi s^2}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\Omega}{\hbar s^2}}\hat{x}\right)\exp\left(-\frac{\Omega}{2\hbar s^2}\hat{x}^2\right). \tag{24}$$

The relation between eigenstates of $\hat{I}$ and $\hat{I}'$ can be written as

$$\langle \hat{x}|n\rangle = \hat{U}\langle \hat{x}|n'\rangle. \tag{25}$$

Applying equation (15) to the above equation gives

$$\langle \hat{x}|n\rangle = \sqrt[4]{\frac{\Omega}{\hbar\pi s^2}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\Omega}{\hbar s^2}}\hat{x}\right)\exp\left[\frac{1}{2\hbar}\left(\frac{\mathrm{i}m\dot{s}}{s}\mathrm{e}^{2\beta t}-\frac{\Omega}{s^2}\right)\hat{x}^2\right]. \tag{26}$$

The $\hat{x}$ space Schrödinger solution $\langle \hat{x}|\psi_n\rangle$ of Hamiltonian, equation (1), is the same as the eigenstate of $\hat{I}$, except for the time-dependent phase factor, $\gamma_n(t)$ [27]:

$$\langle \hat{x}|\psi_n\rangle = \langle \hat{x}|n\rangle \mathrm{e}^{\mathrm{i}\gamma_n(t)}. \tag{27}$$

Substitution of the above equation into the Schrödinger equation gives

$$\hbar\dot{\gamma}_n(t) = \langle \phi_n|\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\hat{H}\right)|\phi_n\rangle. \tag{28}$$

Making use of equation (1), the above equation can be easily solved as

$$\gamma_n(t) = -\frac{\Omega}{m}\left(n+\frac{1}{2}\right)\int_0^t \frac{1}{s^2(t')}\mathrm{e}^{-2\beta t'}\,\mathrm{d}t'. \tag{29}$$

Then, substitution of equations (26) and (29) into equation (27) gives the full wavefunction as

$$\langle \hat{x}|\psi_n\rangle = \sqrt[4]{\frac{\Omega}{\hbar\pi s^2}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\Omega}{\hbar s^2}}\hat{x}\right)\exp\left[\frac{1}{2\hbar}\left(\frac{\mathrm{i}m\dot{s}}{s}\mathrm{e}^{2\beta t}-\frac{\Omega}{s^2}\right)\hat{x}^2\right]$$
$$\times\exp\left[-\mathrm{i}\frac{\Omega}{m}\left(n+\frac{1}{2}\right)\int_0^t \frac{1}{s^2(t')}\mathrm{e}^{-2\beta t'}\,\mathrm{d}t'\right]. \tag{30}$$

This is the exact solution of the Schrödinger equation for the given system. We show the absolute square of this in figure 1 for $n = 1$. We used $c_1 = c_3 = 1$, $c_2 = 0$, $m = 1$ and $\hbar = 1$ for all figures.

We can obtain the variation of $\hat{x}$ and $\hat{p}$ in number state, respectively, as

$$\begin{aligned}\Delta\hat{x} &= [\langle\psi_n|\hat{x}^2|\psi_n\rangle - (\langle\psi_n|\hat{x}|\psi_n\rangle)^2]^{1/2}\\ &= \sqrt{\frac{s^2\hbar}{\Omega}\left(n+\frac{1}{2}\right)},\end{aligned} \tag{31}$$

$$\begin{aligned}\Delta\hat{p} &= [\langle\psi_n|\hat{p}^2|\psi_n\rangle - (\langle\psi_n|\hat{p}|\psi_n\rangle)^2]^{1/2}\\ &= \sqrt{\frac{\hbar\Omega}{s^2}\left[1+\left(\frac{ms\dot{s}}{\Omega}\right)^2\mathrm{e}^{4\beta t}\right]\left(n+\frac{1}{2}\right)}.\end{aligned} \tag{32}$$

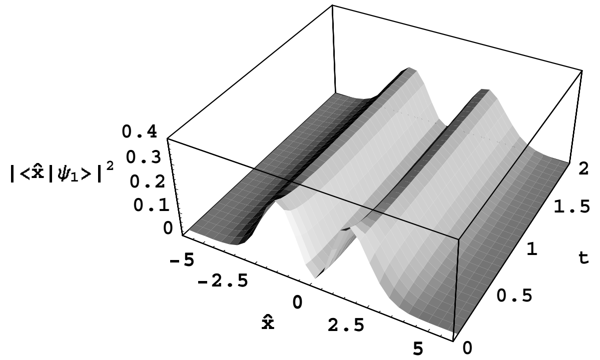

Figure 1. First excited probability density $|\langle\hat{x}|\psi_1\rangle|^2$ as a function of position $\hat{x}$ and time t for $\omega_0 = 0.3$ and $\beta = 0.1$.

By multiplying the above two equations, we can obtain the uncertainty relation

$$\Delta\hat{x}\Delta\hat{p} = \hbar\sqrt{1+\left(\frac{ms\dot{s}}{\Omega}\right)^2 \mathrm{e}^{4\beta t}}\left(n+\frac{1}{2}\right). \tag{33}$$

This differs from the uncertainty relation for the ordinary simple harmonic oscillator and is always larger than $\hbar/2$. The magnitude of the uncertainty relation becomes larger as $\dot{s}$ increases. In other words, if the system varies rapidly with time, it becomes more uncertain.

The uncertainty relation of the q-deformed harmonic oscillator also differs from the uncertainty relation for the simple harmonic oscillator [28]. As the quantum number increases, the uncertainty relation of the time-dependent frequency increases while that of the q-deformed harmonic oscillator decreases.

The mechanical energy of the system can be written as

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\frac{\omega_0}{t}\right)^2 x^2. \tag{34}$$

The nth order of quantum mechanical energy can be obtained by replacing $\dot{x}$ in equation (34) with the corresponding expression containing $\hat{p}$ and averaging with respect to $|\psi_n\rangle$ as

$$E_n(t) = \mathrm{e}^{-4\beta t}\frac{1}{2m}\langle\psi_n|\hat{p}^2|\psi_n\rangle + \frac{1}{2}m\left(\frac{\omega_0}{t}\right)^2\langle\psi_n|\hat{x}^2|\psi_n\rangle. \tag{35}$$

Taking advantage of equation (30), the above equation can be calculated as

$$E_n(t) = \frac{\hbar\Omega}{2m}\left[\frac{1}{s^2}\mathrm{e}^{-4\beta t} + \frac{m^2}{\Omega^2}\left(\dot{s}^2 + \left(\frac{\omega_0}{t}\right)^2 s^2\right)\right]\left(n+\frac{1}{2}\right). \tag{36}$$

Classically, we can define the action integral as $J(t) = \oint p\,\mathrm{d}x$ which is adiabatically an invariant quantity [29]. In the case of this oscillatory system, it is given by $J(t) = 2\pi E(t)\mathrm{e}^{2\beta t}/\omega(t)$. Let us suppose that the classical mechanical energy is exactly known at time t_0. Then, by equating $J(t) = J(t_0)$, we can obtain it at arbitrary time t as

$$E(t) = E(t_0)\frac{t_0}{t}\mathrm{e}^{-2\beta(t-t_0)}. \tag{37}$$

We compare equations (36) and (37) in figure 2. They decrease in a similar fashion, although there is small difference between them.

We consider an ensemble of particles that satisfies the given time-dependent harmonic oscillator motion. Let us assume that these particles conform to a Bose–Einstein distribution

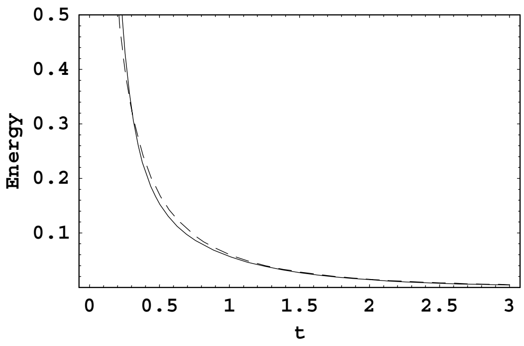

Figure 2. Comparison of quantum (solid curve) and classical (dot curve) energy for $\beta = 0.35$ and $\omega_0 = 0.2$. We used quantum and classical energy given in equations (36) and (37), respectively. We chose $n = 0$ for quantum mechanical energy and $E(t_0) = 0.06$ where $t_0 = 1$ for classical one.

function at equilibrium temperature T. The density operator satisfies the Liouville–von Neumann equation

$$\frac{\partial \hat{\rho}(t)}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{\rho}(t),\ \hat{H}] = 0. \tag{38}$$

We can write the partition function of the system as

$$Z(t) = \sum_{n=0}^{\infty} \langle \psi_n(t)|\mathrm{e}^{-\beta_b \Omega \hat{I}(t)}|\psi_n(t)\rangle, \tag{39}$$

where $\beta_b = 1/(k_b T)$ and k_b is Boltzmann's constant. After performing summation, equation (39) becomes

$$Z(t) = \frac{1}{2\sinh(\beta_b \hbar\Omega/2)}. \tag{40}$$

The density operator of the system in $\hat{x}$ space is given by

$$\hat{\rho}(\hat{x}, \hat{x}', t) = \frac{1}{Z(t)} \sum_{n=0}^{\infty} \langle \hat{x}|\psi_n(t)\rangle \exp\left[-\beta_b \hbar\Omega\left(n + \frac{1}{2}\right)\right] \langle \psi_n(t)|\hat{x}'\rangle. \tag{41}$$

Using equation (30), the above equation can be calculated as

$$\hat{\rho}(\hat{x}, \hat{x}', t) = \left[\frac{\Omega}{\pi\hbar s^2}\tanh\left(\frac{1}{2}\beta_b\hbar\Omega\right)\right]^{1/2} \exp\left\{\frac{\mathrm{i}\dot{s}m}{2\hbar s}\mathrm{e}^{2\beta t}(\hat{x}^2 - \hat{x}'^2) - \frac{\Omega}{4\hbar s^2}\left[(\hat{x} + \hat{x}')^2\tanh\left(\frac{1}{2}\beta_b\hbar\Omega\right) + (\hat{x} - \hat{x}')^2\coth\left(\frac{1}{2}\beta_b\hbar\Omega\right)\right]\right\}. \tag{42}$$

In obtaining the above equation, we used Mehler's formula [30]:

$$\sum_{n=0}^{\infty} \frac{(z/2)^n}{n!} H_n(x) H_n(y) = (1 - z^2)^{-1/2} \exp\left[\frac{2xyz - (x^2 + y^2)z^2}{1 - z^2}\right]. \tag{43}$$

As temperature becomes sufficiently high, equation (42) may be approximated to

$$\hat{\rho}(\hat{x}, \hat{x}', t) \sim \frac{\Omega}{s}\sqrt{\frac{\beta_b}{2\pi}} \exp\left[\frac{\mathrm{i}\dot{s}m}{2\hbar s}\mathrm{e}^{2\beta t}(\hat{x}^2 - \hat{x}'^2) - \frac{1}{2\beta_b\hbar^2 s^2}(\hat{x} - \hat{x}')^2\right]. \tag{44}$$

If the difference between $\hat{x}$ and $\hat{x}'$ is sufficiently small compared with $\sqrt{2\hbar s/(\dot{s}m)}\mathrm{e}^{-\beta t}$ and β_b approaches to zero, equation (44) may be more simply abbreviated to

$$\hat{\rho}(\hat{x}, \hat{x}', t) \sim \beta_b\hbar\Omega\, \delta(\hat{x} - \hat{x}'). \tag{45}$$

On the other hand, at low temperature, equation (42) becomes

$$\hat{\rho}(\hat{x}, \hat{x}', t) \sim \sqrt{\frac{\Omega}{\pi\hbar s^2}} \exp\left[\frac{\mathrm{i}\dot{s}m}{2\hbar s}\mathrm{e}^{2\beta t}(\hat{x}^2 - \hat{x}'^2) - \frac{\Omega}{2\hbar s^2}(\hat{x}^2 + \hat{x}'^2)\right]. \tag{46}$$

The probability that the mass of the oscillator resides at $\hat{x}$ is obtained by taking the diagonal elements of equation (42) as

$$f(\hat{x}) = \left[\frac{\Omega}{\pi\hbar s^2}\tanh\left(\frac{1}{2}\beta_b\hbar\Omega\right)\right]^{1/2} \exp\left[-\frac{\Omega}{\hbar s^2}\tanh\left(\frac{1}{2}\beta_b\hbar\Omega\right)\hat{x}^2\right]. \tag{47}$$

We use this to calculate various expectation values. For high temperature, the above equation reduces to

$$f(\hat{x}) \sim \frac{\Omega}{s}\sqrt{\frac{\beta_b}{2\pi}} \exp\left(-\frac{\beta_b\Omega^2}{2s^2}\hat{x}^2\right). \tag{48}$$

The expectation value of $\hat{x}^2$ in the thermal state may be obtained from the relation

$$\langle\hat{x}^2\rangle = \int_{-\infty}^{\infty} \hat{x}^2 f(\hat{x})\,\mathrm{d}\hat{x}. \tag{49}$$

Using equation (48), the above equation can be calculated as

$$\langle\hat{x}^2\rangle = \frac{\hbar s^2}{2\Omega}\coth\left(\frac{1}{2}\beta_b\hbar\Omega\right). \tag{50}$$

By a similar procedure in $\hat{p}$ space, we can obtain the expectation value of $\hat{p}^2$ as

$$\langle\hat{p}^2\rangle = \frac{\hbar\Omega}{2s^2}\left(1 + \frac{m^2 s^2 \dot{s}^2}{\Omega^2}\mathrm{e}^{4\beta t}\right)\coth\left(\frac{1}{2}\beta_b\hbar\Omega\right). \tag{51}$$

From equation (39), we can confirm that the expectation value of the invariant operator is related to the partition function by the following equation:

$$\langle\hat{I}\rangle\Omega = -\frac{\partial}{\partial\beta_b}\ln Z(t). \tag{52}$$

This expression is exactly the same as that of the energy operator for the ordinary simple harmonic oscillator. Substitution of equation (40) into the above equation gives

$$\langle\hat{I}\rangle = \tfrac{1}{2}\hbar\coth\left(\tfrac{1}{2}\beta_b\hbar\Omega\right). \tag{53}$$

By further evaluation, we easily obtain $\langle\hat{x}\rangle = 0$ and $\langle\hat{p}\rangle = 0$ in the thermal state. The uncertainty relation is given by the following relation:

$$\Delta\hat{x}\Delta\hat{p} = [(\langle\hat{x}^2\rangle - (\langle\hat{x}\rangle)^2)(\langle\hat{p}^2\rangle - (\langle\hat{p}\rangle)^2)]^{1/2}. \tag{54}$$

Taking advantage of equations (50) and (51), the above equation can be calculated as

$$\Delta\hat{x}\Delta\hat{p} = \frac{\hbar}{2}\sqrt{1 + \frac{m^2\dot{s}^2 s^2}{\Omega^2}\mathrm{e}^{4\beta t}}\coth\left(\frac{1}{2}\beta_b\hbar\Omega\right). \tag{55}$$

This relation simply replaces factor $(n + 1/2)$ of equation (33) with $\coth(\beta_b\hbar\omega/2)/2$. Since these factors are constant, the two uncertainty relations have the same time dependency, as expected.

3. Harmonic oscillator of inversely decreasing frequency with inverse quadratic potential

Let us add a singular term to the previous Hamiltonian. Then, we may write the Hamiltonian as

$$\hat{H} = \mathrm{e}^{-2\beta t}\frac{\hat{p}^2}{2m} + \frac{1}{2}\mathrm{e}^{2\beta t}m\omega^2(t)\hat{x}^2 + \mathrm{e}^{-2\beta t}\frac{k}{2m}\frac{1}{\hat{x}^2}, \tag{56}$$

where $\omega(t)$ is the same as in equation (2) and k is a positive constant. Taking advantage of Hamilton's equation of motion, we can derive the classical equation of motion satisfying the above Hamiltonian as

$$\ddot{\hat{x}} + 2\beta\dot{\hat{x}} + \left(\frac{\omega_0}{t}\right)^2\hat{x} - \mathrm{e}^{-4\beta t}\frac{k}{m^2}\frac{1}{\hat{x}^3} = 0. \tag{57}$$

The invariant operator corresponding to the Hamiltonian, equation (56), may be written as

$$\hat{I}(t) = \frac{1}{2\Omega}\left\{\frac{\Omega^2}{s^2(t)}\hat{x}^2 + [m\mathrm{e}^{2\beta t}\dot{s}(t)\hat{x} - s(t)\hat{p}]^2 + \frac{ks^2}{\hat{x}^2}\right\}. \tag{58}$$

We can perform the same procedure as in the previous case using the unitary operator given in equation (15) so that the above equation can be transformed to

$$\hat{I}' = -\frac{\hbar^2 s^2}{2\Omega}\frac{\partial^2}{\partial\hat{x}^2} + \frac{\Omega}{2s^2}\hat{x}^2 + \frac{s^2 k}{2\Omega}\frac{1}{\hat{x}^2}. \tag{59}$$

Making use of the relation given in equation (18), the above equation may be simplified to

$$\hat{I}' = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial\hat{X}^2} + \frac{1}{2}\hat{X}^2 + \frac{k}{2\hat{X}^2}. \tag{60}$$

The eigenstate and eigenvalue of the above equation can be written by [26]

$$\langle\hat{X}|N'\rangle = \sqrt[4]{\frac{4}{\hbar}}\left(\frac{2\Gamma(n+1)}{\Gamma(n+a+1)}\right)^{1/2}\left(\frac{\hat{X}^2}{\hbar}\right)^{(2a+1)/4}L_n^a\!\left(\frac{\hat{X}^2}{\hbar}\right)\exp\left(-\frac{\hat{X}^2}{2\hbar}\right), \tag{61}$$

$$\lambda_n = \hbar(2n+a+1), \tag{62}$$

where L_n^a is the nth-order associated Laguerre polynomial defined in [30] and a is given by

$$a = \frac{1}{2}\left(1 + \frac{4k}{\hbar^2}\right) \tag{63}$$

which is clearly constant. With the same relation given in equation (25), we can obtain the eigenstate of the untransformed invariant operator as

$$\begin{aligned}\langle\hat{x}|n\rangle = {} & \left(\frac{\Omega}{\hbar s^2}\right)^{(a+1)/2}\left(\frac{2\Gamma(n+1)}{\Gamma(n+a+1)}\right)^{1/2}\hat{x}^{a+1/2}L_n^a\left(\frac{\Omega}{\hbar s^2}\hat{x}^2\right)\\ & \times\exp\left[\frac{1}{2\hbar}\left(\frac{\mathrm{i}m\dot{s}}{s}\mathrm{e}^{2\beta t} - \frac{\Omega}{s^2}\right)\hat{x}^2\right].\end{aligned} \tag{64}$$

By the same procedure as in the previous case, the phase factor of the wavefunction can be calculated as

$$\gamma_n(t) = -(2n+a+1)\frac{\Omega}{m}\int_0^t\frac{1}{s^2(t')}\mathrm{e}^{-2\beta t'}\,\mathrm{d}t'. \tag{65}$$

Then, we may write the full wavefunction that satisfies the Schrödinger equation as

$$\begin{aligned}\langle\hat{x}|\psi_n\rangle = &\left(\frac{\Omega}{\hbar s^2}\right)^{(a+1)/2}\left(\frac{2\Gamma(n+1)}{\Gamma(n+a+1)}\right)^{1/2}\hat{x}^{a+1/2}L_n^a\left(\frac{\Omega}{\hbar s^2}\hat{x}^2\right)\\ &\times\exp\left[\frac{1}{2\hbar}\left(\frac{\mathrm{i}m\dot{s}}{s}\mathrm{e}^{2\beta t}-\frac{\Omega}{s^2}\right)\hat{x}^2\right]\\ &\times\exp\left[-\mathrm{i}(2n+a+1)\frac{\Omega}{m}\int_0^t\frac{1}{s^2(t')}\mathrm{e}^{-2\beta t'}\,\mathrm{d}t'\right].\end{aligned} \tag{66}$$

To simplify the problem, let us choose $k=\Omega^2$. Then the expectation value of the Hamiltonian can be calculated as

$$\langle\psi_n|\hat{H}|\psi_n\rangle = \frac{1}{2}\hbar\Omega\left[\frac{1}{ms^2}\mathrm{e}^{-2\beta t}+\frac{m}{\Omega^2}\mathrm{e}^{2\beta t}\left(\dot{s}^2+\left(\frac{\omega_0}{t}\right)^2s^2\right)\right](2n+a+1). \tag{67}$$

The quantum mechanical energy expectation value is given by the relation

$$E_n=\mathrm{e}^{-4\beta t}\frac{1}{2m}\langle\psi_n|\hat{p}^2|\psi_n\rangle+\frac{1}{2}m\left(\frac{\omega_0}{t}\right)^2\langle\psi_n|\hat{x}^2|\psi_n\rangle\mathrm{e}^{-4\beta t}\frac{k}{2m}\langle\psi_n|\frac{1}{\hat{x}^2}|\psi_n\rangle. \tag{68}$$

Making use of equation (66), we can calculate the above equation as

$$E_n=\frac{\hbar\Omega}{2m}\left[\frac{1}{s^2}\mathrm{e}^{-4\beta t}+\frac{m^2}{\Omega^2}\left(\dot{s}^2+\left(\frac{\omega_0}{t}\right)^2s^2\right)\right](2n+a+1). \tag{69}$$

Let us suppose that there is an ensemble of boson particles satisfying the same properties as in the previous section at equilibrium temperature T. Then, using the same relation, equation (39), we can obtain the partition function as

$$Z(t)=\frac{1}{2\mathrm{e}^{\beta_b\hbar\Omega a}\sinh(\beta_b\hbar\Omega)}. \tag{70}$$

The density operator in $\hat{x}$ space is given by

$$\hat{\rho}(\hat{x},\hat{x}',t)=\frac{1}{Z(t)}\sum_{n=0}^{\infty}\langle\hat{x}|\psi_n(t)\rangle\exp\left[-\beta_b\hbar\Omega\,(2n+a+1)\right]\langle\psi_n(t)|\hat{x}'\rangle. \tag{71}$$

Using equation (66), the above equation can be calculated as

$$\hat{\rho}(\hat{x},\hat{x}',t)=\frac{2\Omega}{\hbar s^2}\sqrt{\hat{x}\hat{x}'}\mathrm{e}^{\beta_b\hbar\Omega a}I_a\left[\frac{\Omega}{\hbar s^2\sinh(\beta_b\hbar\Omega)}\hat{x}\hat{x}'\right]\exp[u\hat{x}^2+u^*\hat{x}'^2], \tag{72}$$

where

$$u=\frac{1}{2\hbar s}\left[\mathrm{i}\dot{s}m\mathrm{e}^{2\beta t}-\frac{\Omega}{s}\coth(\beta_b\hbar\Omega)\right]. \tag{73}$$

In the calculation of equation (72), we used the following relation [30]:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+a+1)}L_n^a(x)L_n^a(y)z^n\\ &\quad=(1-z)^{-1}\exp\left(-z\frac{x+y}{1-z}\right)(xyz)^{-a/2}I_a\left[2\frac{(xyz)^{1/2}}{1-z}\right],\quad\text{for }|z|<1,\end{aligned} \tag{74}$$

where $I_a(y)$ is a modified Bessel function of the first kind defined as

$$I_a(y)=\sum_{j=0}^{\infty}\left(\frac{y}{2}\right)^{2j+a}\frac{1}{\Gamma(j+1)\Gamma(j+a+1)}. \tag{75}$$

For high temperature, equation (72) can be approximated to

$$\hat{\rho}(\hat{x}, \hat{x}', t) \sim \frac{2\Omega}{\hbar s^2}\sqrt{\hat{x}\hat{x}'} I_a\left(\frac{\hat{x}\hat{x}'}{\beta_b \hbar^2 s^2}\right)\exp\left[\frac{\mathrm{i}m\dot{s}}{2\hbar s}\mathrm{e}^{2\beta t}(\hat{x}^2 - \hat{x}'^2) - \frac{1}{2\beta_b \hbar^2 s^2}(\hat{x}^2 + \hat{x}'^2)\right]. \tag{76}$$

On the other hand, at low temperature it becomes

$$\hat{\rho}(\hat{x}, \hat{x}', t) \sim \frac{2\Omega}{\hbar s^2}\sqrt{\hat{x}\hat{x}'}\mathrm{e}^{\beta_b \hbar \Omega a} I_a\left[\frac{2\Omega}{\hbar s^2}\mathrm{e}^{-\beta_b \hbar \Omega}\hat{x}\hat{x}'\right] \times \exp\left[\frac{\mathrm{i}m\dot{s}}{2\hbar s}\mathrm{e}^{2\beta t}(\hat{x}^2 - \hat{x}'^2) - \frac{\Omega}{2\hbar s^2}(\hat{x}^2 + \hat{x}'^2)\right]. \tag{77}$$

The diagonal element of the density operator is given by

$$f(\hat{x}) = \frac{2\Omega}{\hbar s^2}\hat{x}\mathrm{e}^{\beta_b \hbar \Omega a} I_a\left(\frac{\Omega}{\hbar s^2 \sinh(\beta_b \hbar \Omega)}\hat{x}^2\right)\exp\left[-\frac{\Omega}{\hbar s^2}\coth(\beta_b \hbar \Omega)\,\hat{x}^2\right]. \tag{78}$$

We can write the expectation value of $\hat{x}$ in the thermal state in the form

$$\langle \hat{x} \rangle = \int_0^\infty \hat{x} f(\hat{x})\,\mathrm{d}\hat{x}. \tag{79}$$

Using equation (78), the above equation can be calculated as

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar s^2}{\Omega}}\mathrm{e}^{\beta_b \hbar \Omega a}\frac{[\tanh(\beta_b \hbar \Omega)]^{3/2}}{[\cosh(\beta_b \hbar \Omega)]^a}\frac{\Gamma(a + 3/2)}{2^a \Gamma(a+1)} \times {}_2F_1\left(\frac{3}{4} + \frac{a}{2}, \frac{4}{5} + \frac{a}{2}, 1 + a, \frac{1}{[\cosh(\beta_b \hbar \Omega)]^2}\right). \tag{80}$$

By a similar procedure, we calculated several other expectation values:

$$\langle \hat{x}^2 \rangle = \frac{\hbar s^2}{2^a \Omega}\mathrm{e}^{\beta_b \hbar \Omega a}\frac{[\tanh(\beta_b \hbar \Omega)]^2}{[\cosh(\beta_b \hbar \Omega)]^a}(1 + a) \times {}_2F_1\left(1 + \frac{a}{2}, \frac{3 + a}{2}, 1 + a, \frac{1}{[\cosh(\beta_b \hbar \Omega)]^2}\right), \tag{81}$$

$$\left\langle \frac{1}{\hat{x}} \right\rangle = \sqrt{\frac{\Omega}{\hbar s^2}}\mathrm{e}^{\beta_b \hbar \Omega a}\frac{[\tanh(\beta_b \hbar \Omega)]^{1/2}}{[\cosh(\beta_b \hbar \Omega)]^a}\frac{\Gamma(a + 1/2)}{2^a \Gamma(a+1)} \times {}_2F_1\left(\frac{1}{4} + \frac{a}{2}, \frac{3}{4} + \frac{a}{2}, 1 + a, \frac{1}{[\cosh(\beta_b \hbar \Omega)]^2}\right), \tag{82}$$

$$\left\langle \frac{1}{\hat{x}^2} \right\rangle = \frac{\Omega}{\hbar s^2}\mathrm{e}^{\beta_b \hbar \Omega a}\frac{1}{2^a a}\frac{1}{[\cosh(\beta_b \hbar \Omega)]^a} \times {}_2F_1\left(\frac{1 + a}{2}, \frac{a}{2}, 1 + a, \frac{1}{[\cosh(\beta_b \hbar \Omega)]^2}\right). \tag{83}$$

4. Summary

Taking advantage of the invariant operator, we obtained the Schrödinger solution for the harmonic oscillator of inversely decreasing frequency with and without inverse quadratic potential. We calculated the uncertainty relation and the quantum mechanical energy expectation value in the number state. By comparing equations (33) and (55), we can confirm that the uncertainty relation in the thermal state varies in the same fashion as that in the number state.

Figure 1 shows that the amplitude of given harmonic oscillator decreases with time depending on the value of damping constant β. The uncertainty relation is always larger than $\hbar/2$ in both the number and thermal state. By differentiating with respect to variables time t and ω_0, we can confirm that this relation does not vary significantly with the values of these variables. The quantum mechanical energy expectation values decrease with time, which is in agreement with the analysis of classical energy. We determined density operators to satisfy the Liouville–von Neumann equation and used them to calculate various expectation values of variables in the thermal state.

References

[1] Moshinsky M and Smirnov Y F 1996 *The Harmonic Oscillator in Modern Physics* (New York: Harwood Academic)
[2] Feynman R P 1951 *Phys. Rev.* **84** 108
[3] Schwinger J 1953 *Phys. Rev.* **91** 728
[4] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer) ch 12 and 17 and references therein
[5] Song D-Y 1999 *Phys. Rev.* A **59** 2616
[6] Yeon K H, Lee K K, Um C I, George T F and Pandey L N 1993 *Phys. Rev.* A **48** 2716
[7] Yeon K H, Kim H J, Um C I, George T F and Pandey L N 1996 *Nuovo Cimento Soc. Ital. Fis.* B **111** 963
[8] Qian S W, Huang B-W and Gu Z-Y 2001 *J. Phys. A: Math. Gen.* **34** 5613
[9] Hartley J G and Ray J R 1982 *Phys. Rev.* A **25** 2388
[10] Yeon K H, Kim D H, Um C I, George T F and Pandey L N 1997 *Phys. Rev.* A **55** 4023
[11] Um C I, Kim I H, Yeon K H, George T F and Pandey L N 1997 *J. Phys. A: Math. Gen.* **30** 2545
[12] Um C I, Kim I H, Yeon K H, George T F and Pandey L N 1996 *Phys. Rev.* A **54** 2707
[13] Um C I, Choi J R and Yeon K H 2001 *Korean J. Phys. Soc.* **38** 447
Um C I, Choi J R and Yeon K H 2001 *Korean J. Phys. Soc.* **38** 452
[14] Brown L S 1991 *Phys. Rev. Lett.* **66** 527
[15] Fu-li Li, Wang S J, Weiguny A and Lin D L 1994 *J. Phys. A: Math. Gen.* **27** 985
[16] Landovitz L F, Levine A M and Schreiber W M 1979 *Phys. Rev.* A **20** 1162
[17] Pedrosa I A, Serra G P and Guedes I 1997 *Phys. Rev.* A **56** 4300
[18] Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
[19] Lewis H R Jr 1968 *J. Math. Phys.* **9** 1976
[20] Ray J R and Reid J L 1979 *Phys. Lett.* A **71** 317
[21] Korsch H J 1979 *Phys. Lett.* A **74** 294
[22] Kaushal R S and Korsch H J 1981 *J. Math. Phys.* **22** 1904
[23] Kaushal R S and Parashar D 1997 *Phys. Rev.* A **55** 2610
[24] Struckmeier J and Riedel C 2000 *Phys. Rev. Lett.* **85** 3830
[25] Kim S P and Lee C H 2000 *Phys. Rev.* D **62** 125020
[26] Calogero F 1969 *J. Math. Phys.* **10** 2191
[27] Lewis H R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[28] Raychev P 1995 *Adv. Quantum Chem.* **26** 239
[29] Goldstein H 1980 *Classical Mechanics* 2nd edn (New York: Addison-Wesley) section 7 of ch 11
[30] Erdély A 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)